Crete, July 2007 Summer School
on **Bose-Einstein Condensation**

Theory of interacting Bose and Fermi gases in traps

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2nd lecture

Dynamics of superfluids

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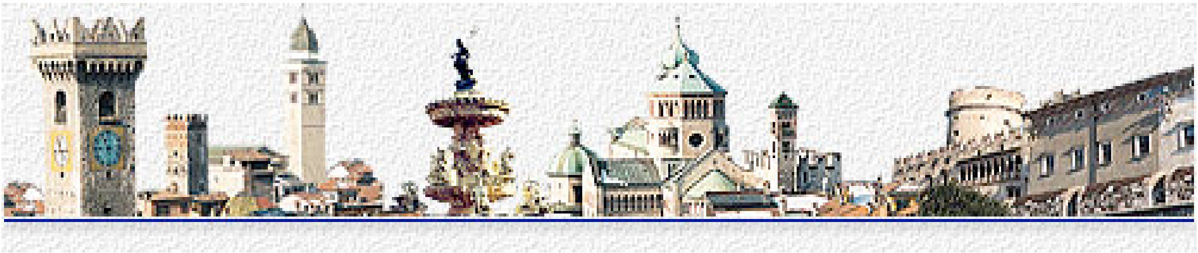


Understanding superfluid features requires theory for transport phenomena (crucial interplay between dynamics and superfluidity)

Macroscopic dynamic phenomena in superfluids
(expansion, collective oscillations, moment of inertia)
are described by theory of **irrotational hydrodynamics**

More microscopic theories required to describe
other superfluid phenomena
(vortices, Landau critical velocity, pairing gap)

HYDRODYNAMIC THEORY OF SUPERFLUIDS

Basic assumptions:

- **Irrotationality** constraint
  (follows from the phase of order parameter)

- **Conservation laws**
  (equation of continuity, equation for the current)

Basic ingredient:

- **Equation of state**

Consequence of Galilean invariance: from the equation for the field operator to the hydrodynamic equations of superfluids

Heisenberg equation for the field operator in uniform systems (Bose field)

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(r,t)=[\hat{\Psi}(r,t),H]=[-\frac{\hbar^2}{2m}\nabla^2+2\int dr'\hat{\Psi}^+(r',t)V(r-r')\hat{\Psi}(r',t)]\hat{\Psi}(r,t)$$

(similar equation for Fermi field operator)

fluid at rest **fluid moving with velocity v**

If $\hat{\Psi}(r,t)$ is solution, $\hat{\Psi}(r-vt,t)\exp[\frac{i}{\hbar}(mrv-\frac{1}{2}mv^2t)]$

is also solution (Galilean transformation with velocity v)

Order parameter $<\hat{\Psi}>$ ($<\hat{\Psi}\hat{\Psi}>$ in Fermi case)

aquires phase $S(r,t)=[mrv-(\frac{1}{2}mv^2+\mu)t]/\hbar$

m ® $2m$
in Fermi case

Gradient of the phase → $v_S = \frac{\hbar}{m} \nabla S$ **Superfluid velocity**

$m \circledR \; 2m$ in Fermi case

IRROTATIONALITY of flow is fundamental feature of superfluids: ⇨

- quenching of **moment of inertia**
- quantization of circulation and **quantized vortices**)

Time derivative of the phase → $\hbar \frac{\partial}{\partial t} S(r,t) = -(\frac{1}{2} m v_S^2 + \mu)$

Relationship for **superfluid velocity** and **equation for the phase** are expected to hold also if order parameter varies **slowly** in space and time as well as in the presence of a smooth external potential. $\boxed{\mu \circledR \; \mu + V_{ext}(r)}$

HYDRODYNAMIC EQUATIONS AT ZERO TEMPERATURE

$$\frac{\partial}{\partial t}n+\nabla(vn)=0$$

$$m\frac{\partial}{\partial t}v+\nabla(\frac{1}{2}mv^2+\mu(n)+V_{ext})=0$$

irrotationality

Hydrodynamic equations of superfluids (T=0)
Closed equations for **density** and superfluid **velocity** field

KEY FEATURES OF HD EQUATIONS OF SUPERFLUIDS

- Have **classical** form (do not depend on Planck constant)
- Velocity field is **irrotational**
- Are equations for the total density (not for the condensate density)
- Should be distinguished from **rotational hydrodynamics**.
- Applicable to **low energy, macroscopic**, phenomena
- Hold for both **Bose** and **Fermi** superfluids
- Depend on **equation of state** $\mu(n)$
  (sensitive to quantum correlations, statistics, dimensionality, ...)
- **Equilibrium** solutions (**v**=0) consistent with **LDA** $\mu(n)+V_{ext}(r)=\mu_0$

What do we mean by **macroscopic, low energy** phenomena ?

BEC superfluids

$$\lambda >> \frac{\hbar}{\sqrt{2mgn}}$$

$$\hbar\omega << gn$$

healing length

BCS Fermi superfluids

size of Cooper pairs

$$\lambda >> \frac{\hbar v_F}{\Delta}$$

$$\hbar\omega << \Delta$$

more restrictive than in BEC

superfluid gap

WHAT ARE THE HYDRODYNAMIC EQUATIONS USEFUL FOR ?

They provide quantitative predictions for

- **Expansion** of the gas follwowing sudden release of the trap

- **Collective oscillations** excited by modulating harmonic trap

Quantities of highest interest from both **theoretical** and **experimental** point of view

- **Expansion** provides information on **release energy**, sensitive to **anisotropy**

- **Collective frequencies** are measurable with **highest precision** and can provide accurate test of **equation of state**

Expansion from anysotropic trap

-Initially the gas is confined in anisotropic trap
in situ **density** profile is **anisotropic** too

$$V_{ext} = m(\omega_z^2 z^2 + \omega_\perp^2 r_\perp^2)/2$$

- What happens after release of the trap ?

Non interacting gas expands isotropically

- For large times **density** distribution become **isotropic**
- Consequence of **isotropy** of **momentum distribution**
- holds for ideal Fermi gas and ideal Bose gas above T_C
- **Ideal BEC gas** expands **anysotropically** because momentum distribution of condensate is anysotropic

Superfluids expand anysotropically

Hydrodynamic equations can be solved
after switching off the external potential

For **polytropic** equation of state $\mu \propto n^\gamma$
(holding for unitary Fermi gas ($\gamma = 2/3$) and in BEC gases ($\gamma = 1$)),
and harmonic trapping, **scaling** solutions are available in the form
(Castin and Dum 1996; Kagan et al. 1996)

$$n(x,y,z,t) = (b_x b_y b_z)^{-1} n_0(x/b_x, y/b_y, z/b_z) =$$

$$v(r,t) = \frac{1}{2}\nabla\left(\alpha_x x^2 + \alpha_y y^2 + \alpha_z z^2\right)$$

with the scaling factors b_i satisfying the equation

$$\ddot{b}_i = \omega_i^2 \frac{1}{b_i (b_x b_y b_z)^\gamma}$$

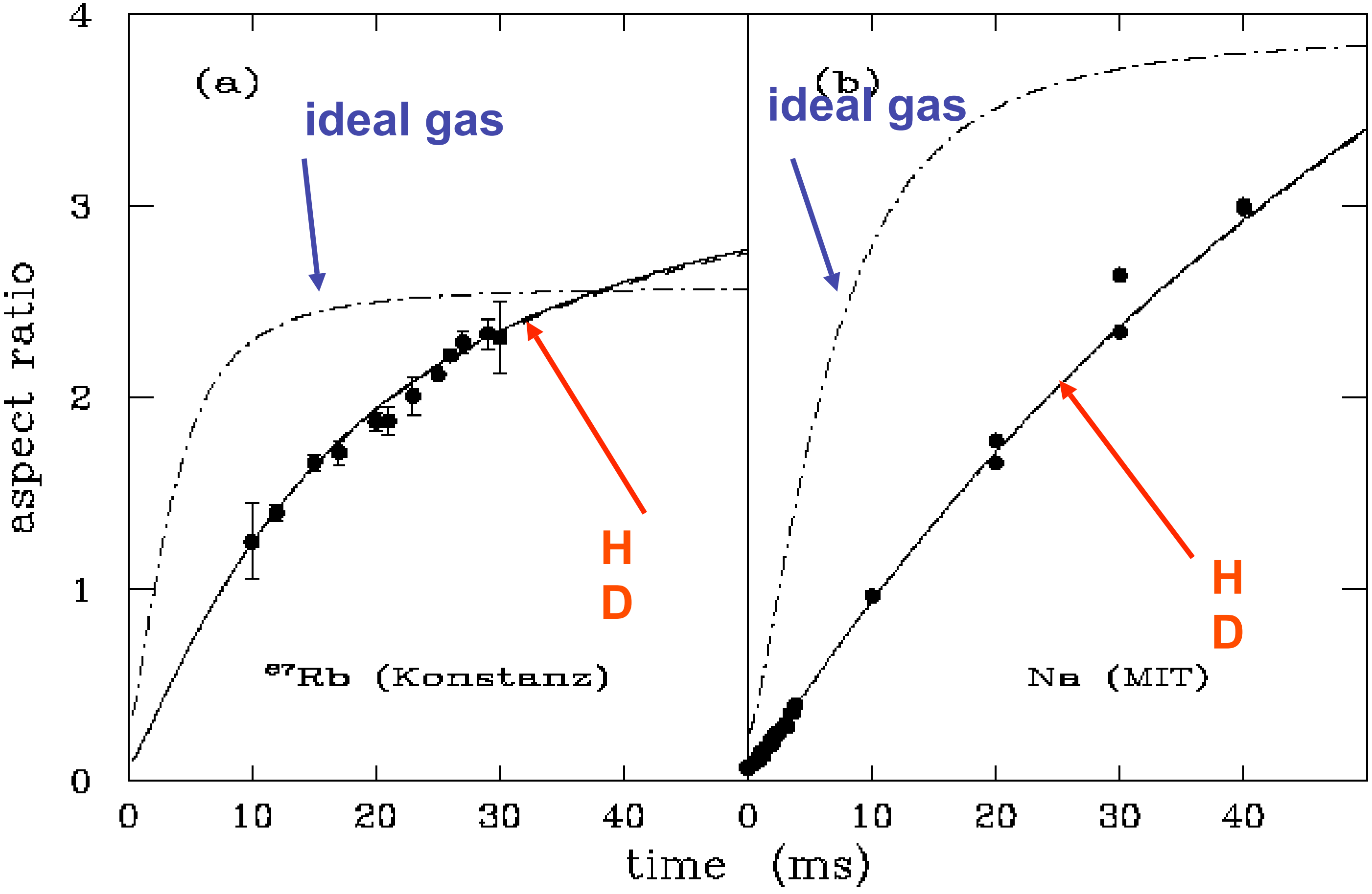
and

$$b_i(t=0) = 1\,,\ \alpha_i = \dot{b}_i / b_i$$

- Expansion **inverts deformation** of density distribution, being faster in the direction of larger density gradients (radial direction in cigar traps)
- expansion transforms **cigar** into **pancake**, and viceversa.

Bosons

Expansion of BEC gases

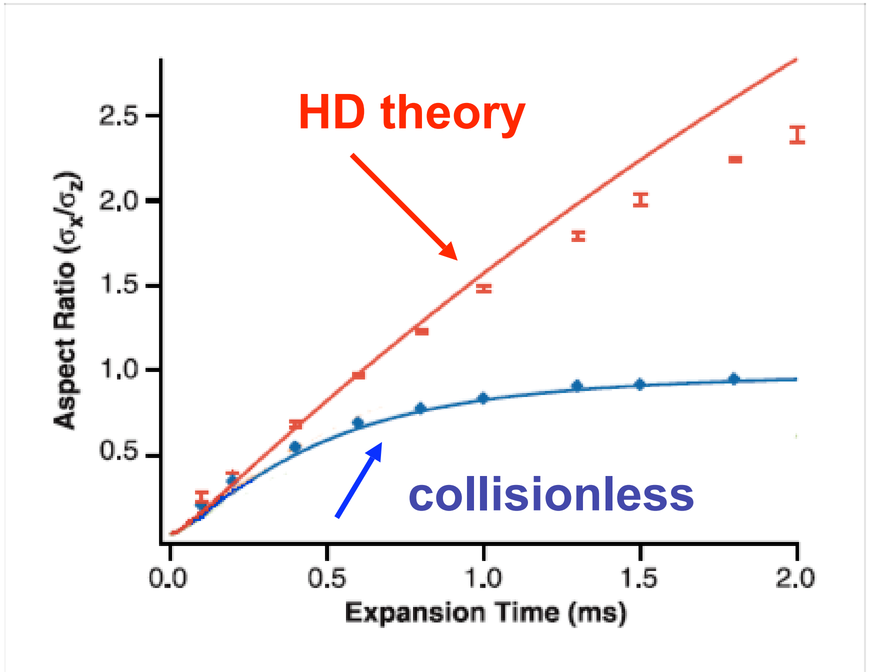


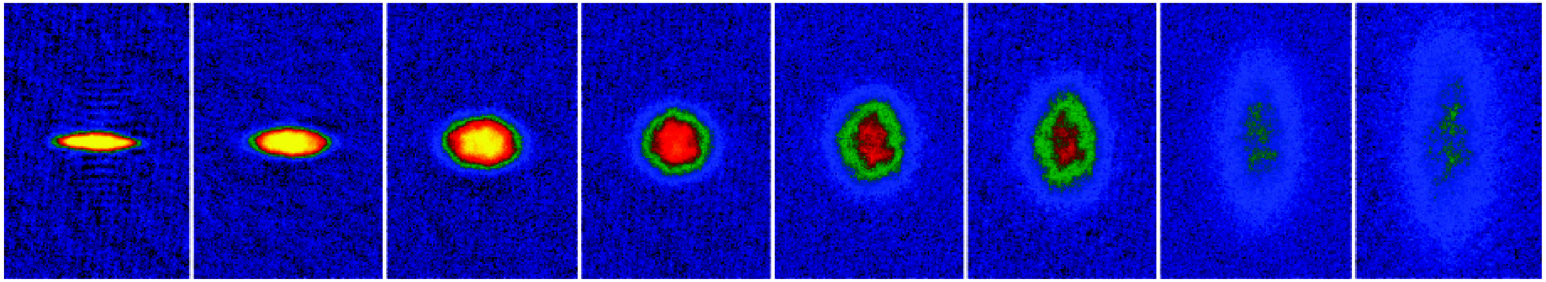
Experiments probe HD nature of the expansion with high accuracy. Aspect ra $\equiv R_{\perp} / Z$

EXPANSION OF ULTRACOLD FERMI GAS

Hydrodynamics predicts anisotropic expansion in Fermi superfluids (Menotti et al,2002)

First experimental evidence for hydrodynamic anisotropic expansion in ultra cold Fermi gas (O'Hara et al, 2002)

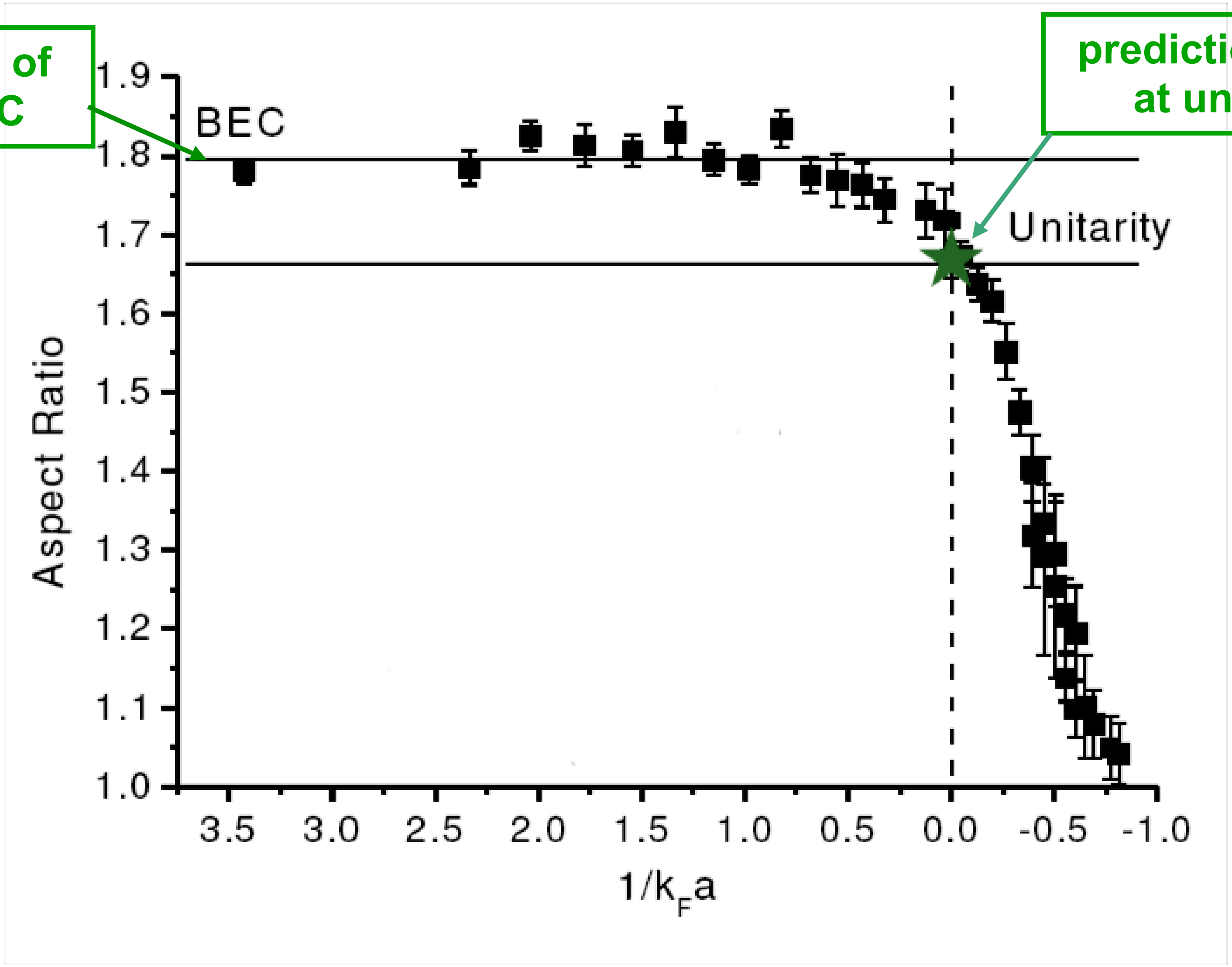




Fermions

Measured aspect ratio after expansion along the BCS-BEC crossover of a Fermi gas

(R. Grimm et al., 2007)



prediction of HD for BEC

prediction of HD at unitarity

Fermions

- *Expansion follows* ***HD*** *behavior on* ***BEC*** *side of the resonance and at* ***unitarity****.*

- *on* ***BCS*** *side it behaves more and more like in non interacting gas (asymptotic isotropy)*

Explanation*:*

- *on* ***BCS*** *side superfluid* ***gap*** *becomes soon* ***exponentially small*** *during the expansion and superfluidity is lost.*

- *At* ***unitarity*** *gap instead always remains of the order of Fermi energy and hence* ***pairs*** *are* ***not easily broken*** *during the expansion*

Collective oscillations in trapped gases

Collective oscillations: unique tool to explore consequence of ***superfluidity*** *and test the* ***equation of state*** *of interacting quantum gases (both* ***Bose*** *and* ***Fermi****)*

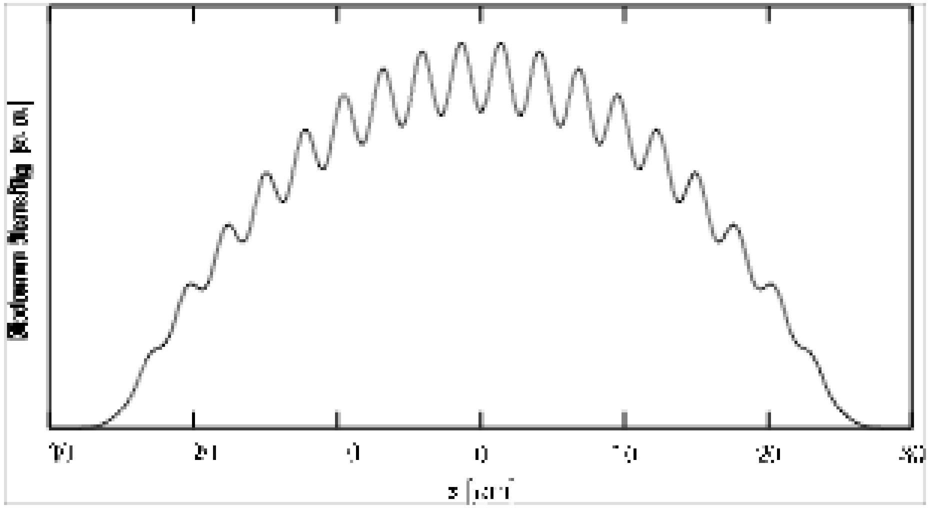
Experimental data *for collective frequencies are available with* ***high precision***

Propagation of sound in trapped gases

In **uniform** medium HD theory gives sound wave solution

$\delta n \propto e^{i(qz-\omega t)}$ with $\omega = cq\ ;\ \ mc^2 = \partial\mu / \partial n$

In **trapped** gases sound waves can propagate if wave length is smaller than axial size of the condensate. Condition is easily satisfied in elongated condensates.



Propagation of sound in elongated traps

-If **wave length** is **larger** than **radial size** of elongated trapped gas sound has **1D** character

$$mc_{1D}^2 = n_1 \partial\mu_1 / \partial n_1$$

where $n_1 = \int n\,dxdy$ and n is determined by TF eq. $\mu(n) + V_{ext}(\vec{r}) = \mu_1$

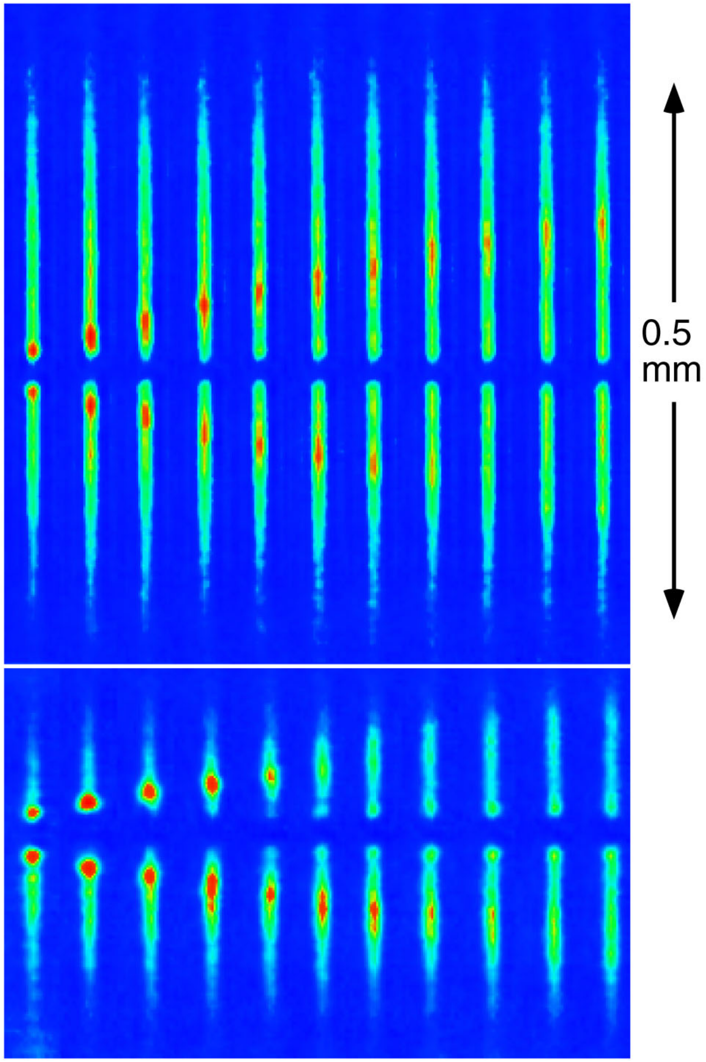
one finds

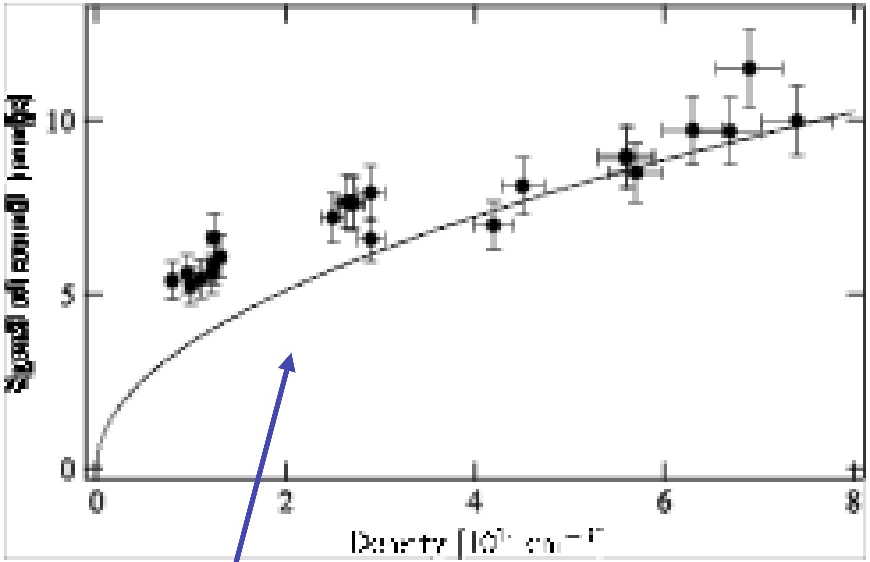
$$mc_{1D}^2 = \frac{\int n\,dxdy}{\int (\partial\mu/\partial n)^{-1}\,dxdy}$$

For BEC gas ($\mu \propto n$) $c_{1D} = c_{bulk}/\sqrt{2}$ *(Zaremba, 1998)*

For unitary Fermi gas ($\mu \propto n^{2/3}$) $c_{1D} = c_{bulk}\sqrt{3/5}$ *(Capuzzi et al, 2006)*

Sound wave packets propagating in a BEC (Mit 97)

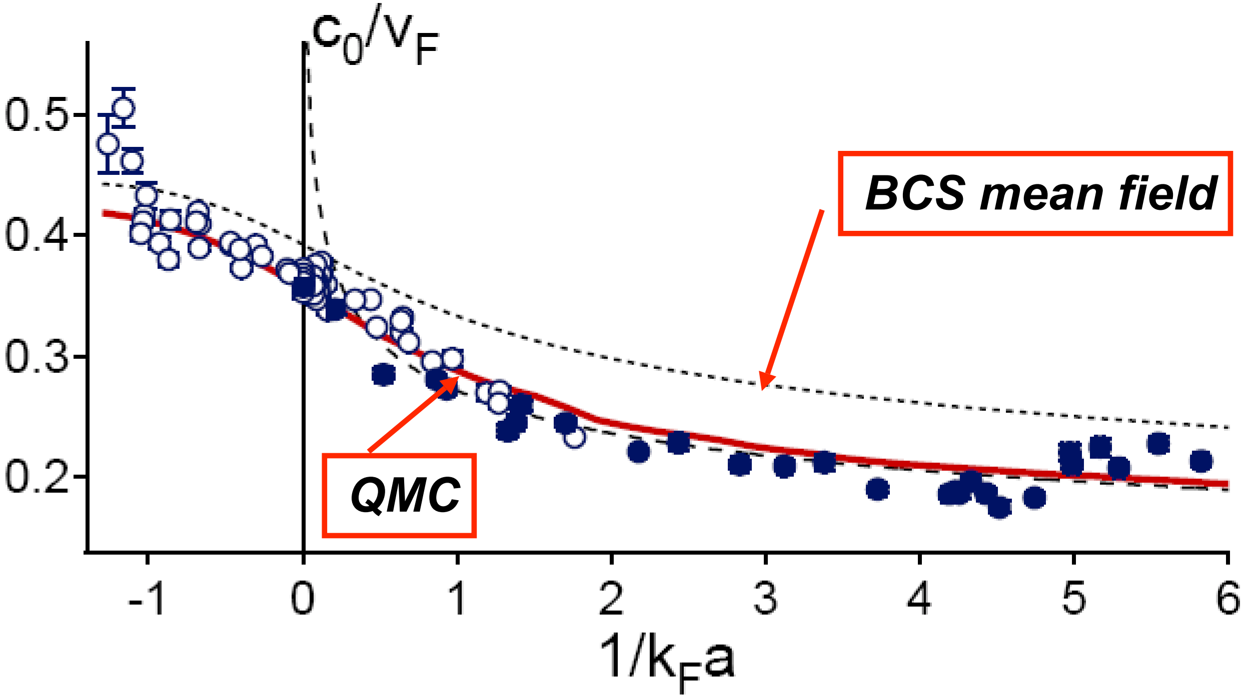




velocity of sound as a function of central density

Fermions

Sound wave packets propagating in an Interacting Fermi gas (Duke, 2006) behavior along the crossover



BCS mean field

QMC

Difference bewteen BCS and QMC reflects:

-at unitarity: different value of β *in eq. of state* $\mu \propto (1+\beta)n^{2/3}$

-On BEC side different molecule-molecule scattering length

Collective oscillations in harmonic trap

When wavelength is of the order of the size of the atomic cloud sound is no longer a useful concept. Solve linearized 3D HD equations

$$\frac{\partial^2}{\partial t^2}\delta n = \nabla(n_0\nabla(\frac{\partial\mu}{\partial n}\delta n))$$

where $n_0(\vec{r})$ *is non uniform equilibrium Thomas Fermi profile*

$$m\frac{\partial}{\partial t}v = -\nabla(\frac{\partial\mu}{\partial n}\delta n)$$

Solutions of HD equations in harmonic trap predict both **surface** and **compression** modes (first investigated in dilute BEC gases (Stringari 96)

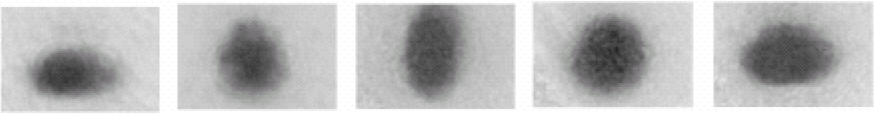
l

Surface modes

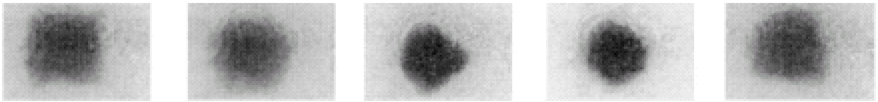
- **Surface** modes are **unaffected** by equation of state
- For isotropic trap one finds $\boxed{\omega = \sqrt{l}\omega_{ho}}$ where $\boxed{l}$ is angular momentum
- surface mode is driven by external potential, **not by surface tension**
- Dispersion law differs from ideal gas value $\boxed{\omega = l\omega_{ho}}$ (**interaction effect**)

Bosons

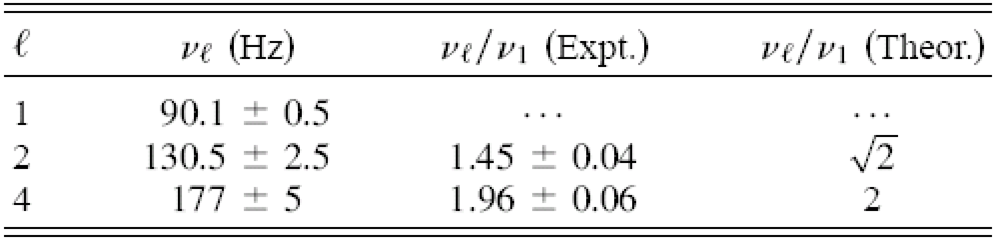
Surface modes in BEC's, Mit 2000



m=2

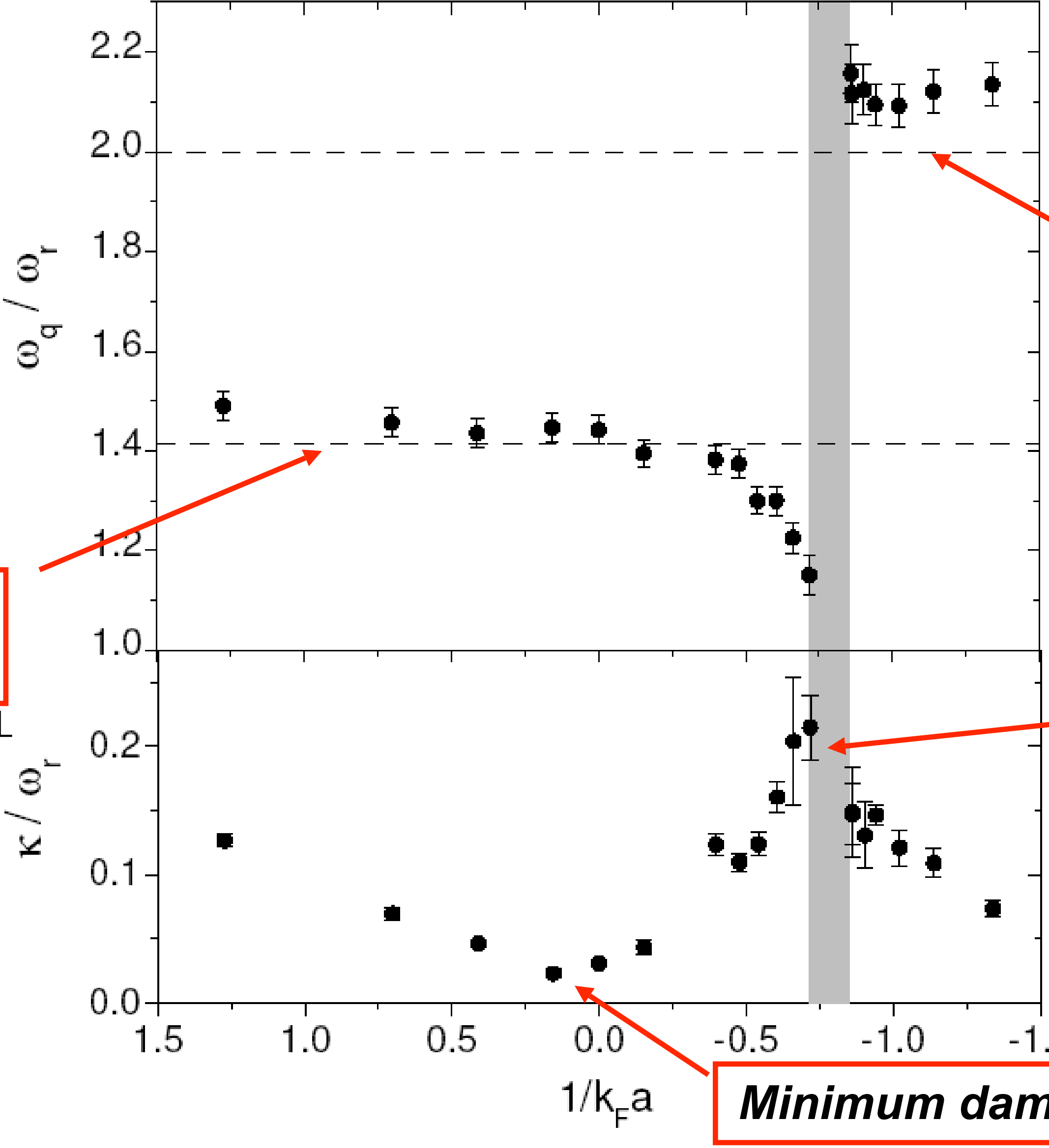


m=4



ℓ	ν_ℓ (Hz)	ν_ℓ/ν_1 (Expt.)	ν_ℓ/ν_1 (Theor.)
1	90.1 ± 0.5	...	...
2	130.5 ± 2.5	1.45 ± 0.04	$\sqrt{2}$
4	177 ± 5	1.96 ± 0.06	2

l=2 Quadrupole mode measured on ultracold Fermi gas along the crossover (Altmeyer et al. 2007)



Ideal gas value

HD prediction

$$\omega = \sqrt{2}\omega_\perp$$

Enhancement of damping

Minimum damping near unitarity

Fermions

- ***Experiments on collective oscillations show that on the BCS side of the resonance superfluidity is broken for relatively small values of*** $1/k_F|a|$ ***(where gap is of the order of radial oscillator frequency)***

- ***Deeper in BCS regime frequency takes collisionless value***

- ***Damping is minimum near resonance***

Compression modes

- Sensitive to the **equation of state**

-analytic solutions for collective frequencies available for **polytropic** equation of state $\mu \propto n^{\gamma}$

- Example: **radial compression** mode in cigar trap $\omega = \sqrt{2(\gamma+1)}\omega_{\perp}$

- At unitarity ($\gamma = 2/3$) one predicts universal value $\omega = \sqrt{10/3}\omega_{\perp} \approx 1.83\omega_{\perp}$

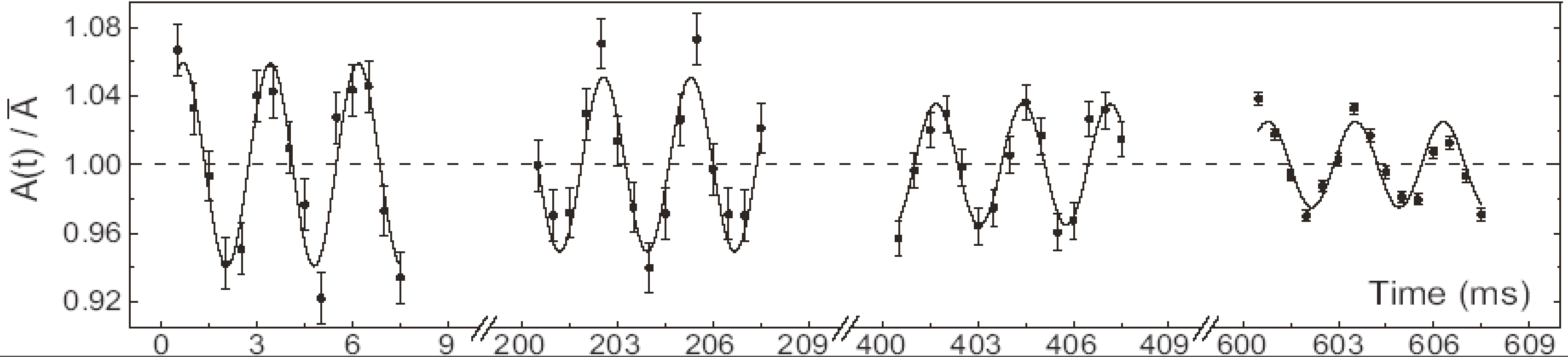
- For a BEC gas one finds $\omega = 2\omega_{\perp}$

Bosons

m=0 radial compression mode at T=0 (Ens 2001)

exp: $\omega = 2.07\,\omega_z$

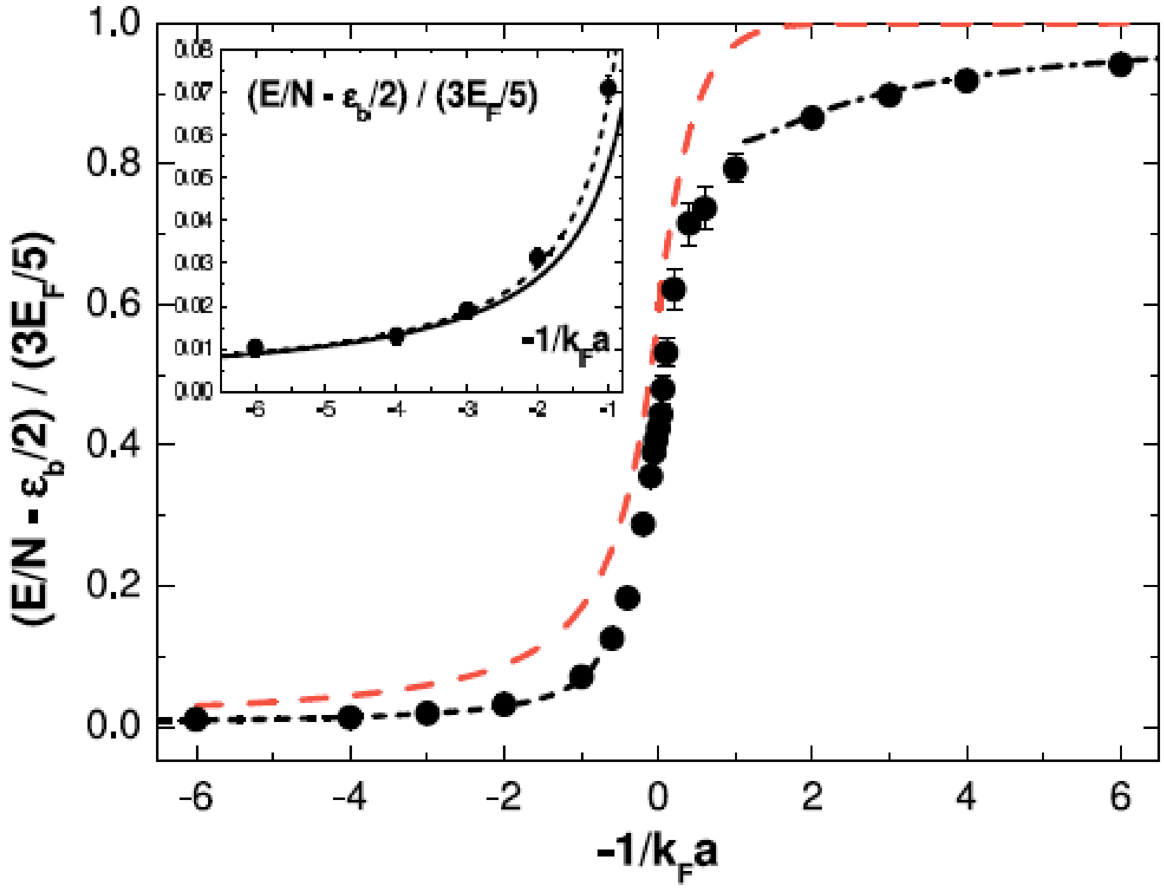
theory: $\omega = 2\,\omega_z$



Fermions

Equation of state along BCS-BEC crossover

- Fixed Node Diffusion MC (Astrakharchick et al., 2004)
- Comparison with mean field BCS theory (- - - - -)

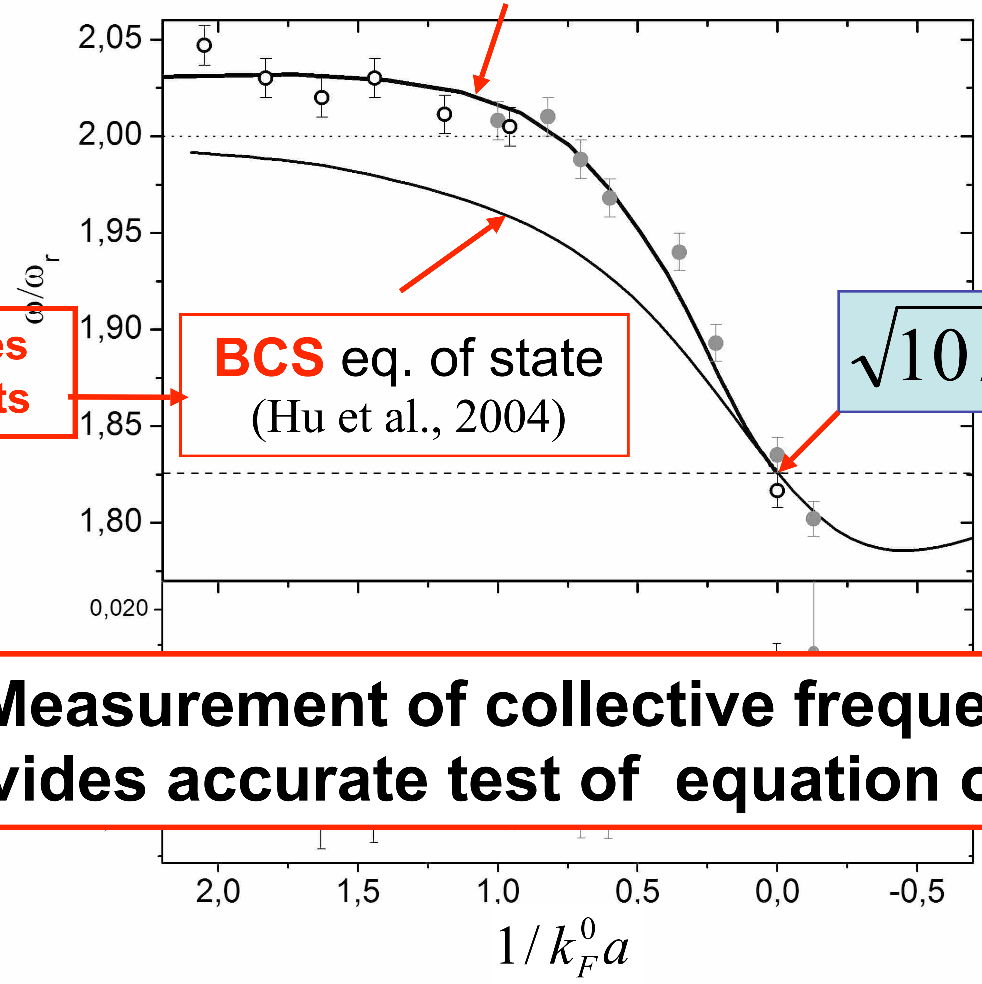


Fermions

Radial breathing mode at Innsbruck (Altmeyer et al., 2007)

MC equation of state (Astrakharchick et al., 2005)

includes beyond mf effects



does not includes beyond mf effects

BCS eq. of state (Hu et al., 2004)

$$\sqrt{10/3} = 1.83$$

universal value at unitarity

Measurement of collective frequencies provides accurate test of equation of state !!

Main conclusions concerning the m=0 radial compression mode in superfluid Fermi gases

- **Accurate confirmation** of the **universal HD value** $\sqrt{10/3}\omega_\perp$ predicted at unitarity.

- Accurate confirmation of **QMC equation of state** on the BEC side of the resonance.

- **First evidence for Lee Huang Lee effect** (enhancement of frequency with respect to BEC value (role of quantum fluctuations)

- LHY effect very sensitive to **thermal effects** (thermal fluctuations prevail on quantum fluctuations except at very low temperature)

Landau's critical velocity

While in BEC gas sound velocity provides critical velocity,
in a Fermi BCS superfluid critical velocity is fixed
by pair breaking mechanisms (role of the ***gap****)*

Landau’s critical velocity

$$v_{cr} = \min_p \frac{\varepsilon(p)}{p}$$

Dispersion law of elementary excitations

- **Landau’s criterion for superfluidity** (**metastability**):
  fluid moving with velocity **smaller** than critical velocity cannot decay (**persistent current**)

- Ideal Bose gas and ideal Fermi gas one has $v_{cr} = 0$

- In interacting Fermi gas one predicts two limiting cases:

BEC (Bogoliubov dispersion)

$$v_{cr} = c \propto \sqrt{a}$$

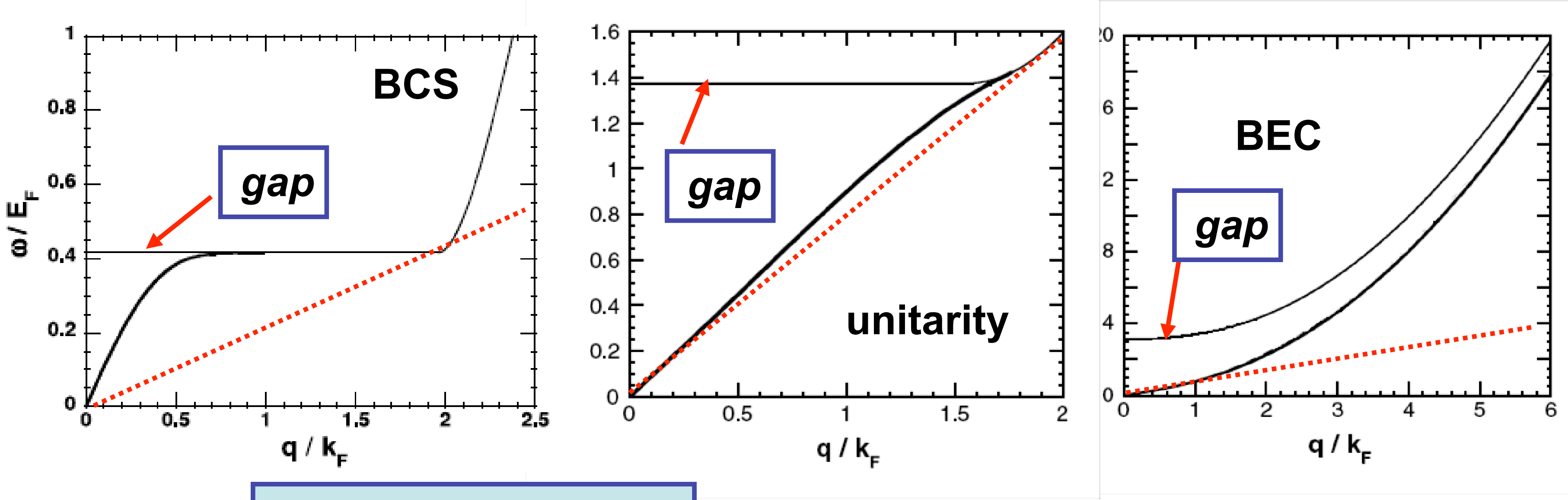
(sound velocity)

BCS (role of the gap)

$$v_{cr} = \Delta / p_F \propto \exp(\pi / 2k_F a)$$

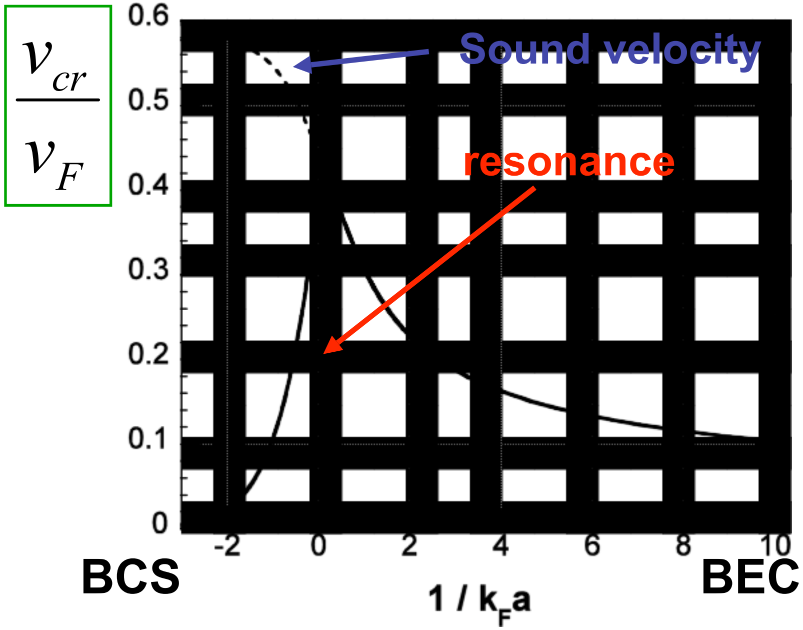
Fermions

Dispersion law along BCS-BEC crossover



Critical velocity

$$\frac{v_{cr}}{v_F}$$



(Combescot, Kagan and Stringari 2006)

Landau's critical velocity is highest near unitarity !!

Some conclusions

- Order parameter: basic ingredient characterizing superfluid behavior (consequence of long range order)

- Microscopic theory available for dilute Bose gas (GP equation). Approximate theories available for Fermi gases (ex: BdG equation)

- Modulus of order parameter directly measurable in dilute Bose gas (coincides with sqrt of density distribution at T=0). In Fermi superfluid order parameter not easily experimentally accessible (some information available through rf transition)

- Phase of order parameter: basic ingredient for dynamic theory of superfluids.

- Systematic confirmation of superfluid dynamic behavior available from the study of collective oscillations

General reviews on BEC and Fermi superfluidity

- **Theory of Bose-Einstein Condensation in trapped gases**
  F. Dalfovo et al., Rev. Mod. Phys. **71**, 463 (1999)

- **Bose-Einstein Condensation in Dilute Gases**
  C. Pethick and H. Smith (Cambridge 2001)

- **Bose-Einstein Condensation in the Alkali Gases: Some Fundamental concepts**
  A. Leggett, Rev. Mod. Phys. **73**, 333 (2001)

- **Bose-Einstein Condensation**
  L. Pitaevskii and S. Stringari (Oxford 2003

- **Ultracold Atomic Fermi gases**
  Proceedings of 2006 Varenna Summer School
  M. Inguscio, W. Ketterle, and Ch. Salomon (in press)

- **Theory of Ultracold Fermi gases**
  S. Giorgini et al. cond-mat/0706.3360 (submitted to Rev.Mod.Phys.)